Problem Sheet Unit 1: Part 2

Random Variable and Probability Density Function; Joint Probability Distribution; & Mathematical expectancy

Q1. Determine the value c so that each of the following functions can serve as a probability distribution of the discrete random variable X:

(a) $f(x) = c(x^2 + 4)$, for $x = 0, 1, 2, 3$;

(b) $f(x) = c\binom{2}{x}\binom{3}{3-x}$, for $x = 0, 1, 2.$

Q2. The shelf life, in days, for bottles of a certain prescribed medicine is a random variable having the density function

$$f(x) = \begin{cases} \frac{20{,}000}{(x+100)^3}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that a bottle of this medicine will have a shell life of
(a) at least 200 days;
(b) anywhere from 80 to 120 days.

Q3. An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond, is

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \le t < 3, \\ \frac{1}{2}, & 3 \le t < 5, \\ \frac{3}{4}, & 5 \le t < 7, \\ 1, & t \ge 7, \end{cases}$$

find
(a) $P(T = 5)$;
(b) $P(T > 3)$;
(c) $P(1.4 < T < 6)$;
(d) $P(T \le 5 \mid T \ge 2)$.

Q4. A shipment of 10 television sets contains 5 defective sets. A hotel makes a random purchase of 3 of the sets. If x is the number of defective sets purchased by the hotel, find the probability distribution of X. Express the results graphically as a probability histogram and construct a graph of the cumulative distribution function.

Find

a) $P(X<2)$
b) $P(1 \le X \le 3)$
c) Expected number of defective sets.

Q5. A continuous random variable X that can assume values between x = 2 and x = 5 has a density function given by $f(x) = 2(1+x)/27$. Find

(a) $P(X < 4)$;
(b) P(3 "X <4).
(c) $E(X)$
(d) $E(X-\mu)^2$

Q6. A cereal manufacturer is aware that the weight of the product in the box varies slightly from box to box. In fact, considerable historical data have allowed the determination of the density function that describes the probability structure for the weight (in ounces). Letting X be the random variable weight, in ounces, the density function can be described as

$$f(x) = \begin{cases} 3x^{-4}, & x > 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that this is a valid density function.
(b) Determine the probability that the weight is smaller than 24 ounces.
(c) The company desires that the weight exceeding 26 ounces be an extremely rare occurrence. What is the probability that this rare occurrence does actually occur?
(d) find the values of $E(X)$ & $E(X-\mu)^2$

Q7. Measurements of scientific systems are always subject to variation, some more than others. There are many structures for measurement error, and statisticians spend a great deal of time modeling these errors. Suppose the measurement error X of a certain physical quantity is decided by the density function

$$f(x) = \begin{cases} k(3 - x^2), & -1 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine k that renders f(x) a valid density function.
(b) Find the probability that a random error in measurement is less than 1/2.
(c) For this particular measurement, it is undesirable if the magnitude of the error (i.e., $|x|$) exceeds 0.8. What is the probability that this occurs?

Q8. Let X denote the reaction time, in seconds, to a certain stimulus and Y denote the temperature (ºF) at which a certain reaction starts to take place. Suppose that two random variables X and Y have the joint density

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{elsewhere}. \end{cases}$$

Find
(a) $P(0 \le X \le \frac{1}{2} \text{ and } \frac{1}{4} \le Y \le \frac{1}{2})$;
(b) $P(X < Y)$.

Q9. Let X denote the number of times a certain numerical control machine will malfunction: 1, 2, or 3 times on any given day. Let Y denote the number of times a technician is called on an emergency call. Their joint probability distribution is given as

$f(x,y)$		$x=1$	$x=2$	$x=3$
y	1	0.05	0.05	0.10
	3	0.05	0.10	0.35
	5	0.00	0.20	0.10

(a) Evaluate the marginal distribution of X.
(b) Evaluate the marginal distribution of Y.
(c) Find $P(Y = 3 \mid X = 2)$.

(d) μ_x & μ_y
(e) σ_x^2 & σ_y^2

Q10. A tobacco company produces blends of tobacco, with each blend containing various proportions of Turkish, domestic, and other tobaccos. The proportions of Turkish and domestic in a blend are random variables with joint density function (X = Turkish and Y = domestic)

$$f(x,y) = \begin{cases} 24xy, & 0 \le x, y \le 1,\ x + y \le 1, \\ 0, & \text{elsewhere}. \end{cases}$$

(a) Find the probability that in a given box the Turkish tobacco accounts for over half the blend.
(b) Find the marginal density function for the proportion of the domestic tobacco.
(c) Find the probability that the proportion of Turkish tobacco is less than 1/8 if it is known that the blend contains 3/4 domestic tobacco.

Q11. A large industrial firm purchases several new word processors at the end of each year, the exact number depending on the frequency of repairs in the previous year. Suppose that the number of word processors, X, purchased each year has the following probability distribution:

x	0	1	2	3
$f(x)$	1/10	3/10	2/5	1/5

If the cost of the desired model is \$1200 per unit and at the end of the year a refund of 50X2 dollars will be issued, how much can this firm expect to spend on new word processors during this year?

Q12. Suppose that X and Y have the following joint probability function:

$f(x,y)$		$x=2$	$x=4$
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

(a) Find the expected value of $g(X, Y) = XY^2$.
(b) Find μ_X and μ_Y .

Q13. Let X and Y be random variables with joint density function

$$f(x,y) = \begin{cases} 4xy, & 0 < x,\ y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $Z = \sqrt{X^2 + Y^2}$.

(a) μ_x & μ_y
(b) σ_x^2 & σ_y^2

Q14. Life-testing results on a first generation microprocessor-based (computer- controlled) toaster indicate that X, the life-span (in years) of the central control chip, is a random variable that is reasonably well-modelled by the pdf:

$$f(x) = \frac{1}{\beta} e^{-x/\beta}\ ; \qquad x > 0$$

with $\beta = 6.25$. A malfunctioning chip will have to be replaced to restore proper toaster operation. The warranty for the chip is to be set at x_w years (in whole integers) such that no more than 15% would have to be replaced before the warranty period expires. Find x_w.

Q15. Given the joint probability density function:

$$f(x_1, x_2) = \begin{cases} ce^{-(x_1+x_2)}; & 0 < x_1 < 1; 0 < x_2 < 2; \\ 0; & \text{elsewhere} \end{cases}$$

First obtain c, then obtain the marginal pdfs $f_1(x_1)$ and $f_2(x_2)$, and hence determine whether or not X_1 and X_2 are independent.